ON THE SOLUTION OF VARIATIONAL PROBLEMS OF SUPERSONIC GAS DYNAMICS

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In solving variational problems of supersonic gas dynamics by the method of Guderly and Armitage [1], it is essential to have linear dependence of the coefficients of the flow parameters being varied on the closing characteristic. This permitted the solution of a number of variational problems [1 to 4] without having to consider the relations between the indicated variations, which arise from the equations of the characteristics. Some problems, however cannot be solved in this manner. In general, therefore, it is also necessary to include the relations on the closing characteristics in the auxiliary functional. This is illustrated by the example below, in constructing the rear part of a minimum drag body with restriction on the length, when the contour may contain an end wall. Two cases are studied. In the first, the pressure on the end wall does not depend on the shapes of the desired contour. Here the Lagrange multipliers, introducing the relations on the characteristics, turn out to be zero, and the solution agrees with that obtained earlier [3]. In the second case, the pressure on the end wall is determined by a condition of the type of Korst's condition [5] and consequently, it depends on the shapes of the desired contour. Here it is necessary to introduce the relations on the closing characteristic. This example is also interesting in that for its solution, questions are considered which are connected with varying the position of the junction of the end wall and the segment of the two-sided extremum. The latter are important for the solution of other problems, e.g. in the construction of the nose part of minimum drag bodies.

1. Let x, y be rectangular coordinates (in axisymmetric case, the x-axis is the axis of symmetry from left to right) ρ the density, p the pressure, u, v the x, y components of the velocity, c the speed of sound, $\nu = 0$ or 1 for plane and axi-symmetric cases respectively. For independent variables, we take y and the stream function ψ, defined by Equation

$$d\psi = y^\nu \rho\,(-vdx + udy)$$

Equilibrium flow of an ideal gas is described by Equations

$$L_1 \equiv \frac{\partial u}{\partial y} - \frac{\partial y^\nu p}{\partial \psi} = 0, \qquad L_2 \equiv \frac{\partial\,(y^\nu \rho v)^{-1}}{\partial y} + \frac{\partial\,(u / v)}{\partial \psi} = 0 \tag{1.1}$$

$$L_3 \equiv \frac{\partial x}{\partial y} - \frac{u}{v} = 0 \qquad \begin{matrix}(1.1)\\ \text{cont.}\end{matrix}$$

Here p, ρ and c are known functions of w and ψ, thus

$$(\partial p \,/\, \partial w)_\psi = -\rho w, \qquad (\partial \rho \,/\, \partial w)_\psi = -c^{-2}\rho w, \qquad w = \sqrt{u^2 + v^2}$$

For supersonic flow ($w > c$), system (1.1) possesses two families of real characteristics. The equations of the characteristics of the second family, needed later on, are

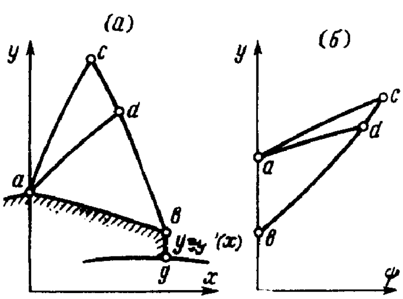
$$L_4 \equiv \frac{d(v/u)}{d\psi} - P\frac{dp}{d\psi} + y^{-(\nu+1)}Q = 0, \qquad L_5 \equiv \frac{dy}{d\psi} + y^{-\nu}Y = 0$$
$$L_6 \equiv \frac{dx}{d\psi} + y^{-\nu}X = 0 \qquad (1.2)$$

Here $d/d\psi$ is the total derivative with respect to ψ, taken along the characteristic;

$$P = P(\psi, u, v) = \frac{\sqrt{w^2 - c^2}}{\rho c u^2}$$
$$Q = Q(\psi, u, v) = \frac{\nu v}{\rho u^2}$$
$$Y = Y(\psi, u, v) = \frac{v\sqrt{w^2 - c^2} - cu}{\rho c w^2}$$
$$X = X(\psi, u, v) = \frac{u\sqrt{w^2 - c^2} + cv}{\rho c w^2}$$

All the parameters are conveniently taken to be dimensionless.

2. Let us consider the problem of constructing the contour ag of the rear part of a body, which gives a minimum wave drag χ for a given flow to the left of ac a specified maximum allowable length of the contour (which we shall take as characteristic dimension), and satisfying a certain condition, (In Fig.1a, ac and ad are characteristics of the first family, bc is a characteristic of the second family). The isoperimetric condition is defined by specifying one of the characteristics of the desired body, e.g., its volume, or its side area, etc. In addition we require that the tail end point of the contour lie on the line $y = y^\circ(x)$, which in particular may be the axis of symmetry.



The direction of the specified contour at the left of point a and that determined by the solution of the variational problem are different. We restrict ourselves to the case when the angle at a is convex. Then for a small change of the contour ag, only the position of the characteristic ad (which bounds the expansion fan) and the flow parameters to its right will change.

The contour ag may consist of a section of a two-sided extremum ab i.e. the end wall bg. If the parameters are denoted by the indices at the corresponding points, and the coordinate system is chosen such that $x_a = 0$, then the equation of the end wall will be $x = 1$ for $y_g \leq y \leq y_b$. The pressure on bg, which will be assumed to be independent of y, will be denoted by p°.

Up to within a nonessential multiplier

$$\chi = \int_a^b y^\nu p\, dy + \int_b^g y^\nu p^\circ\, dy \tag{2.1}$$

The isoperimetric condition has the form

$$K = \int_a^b f(y, x, x')\, dy + \int_b^g f^\circ(y, x, x')\, dy \tag{2.2}$$

where K is a specified constant, f and f° are known functions, and prime denotes the derivative $(\partial / \partial y)_{\psi=\psi_a=0}$.

Thus, it is required to construct the contour $x = x(y)$ where $0 \leq x \leq 1$, $x_a = 0$, $y_g = y^\circ(x_g)$, for which the functional (2.1) attains a minimum, for a specified flow on the characteristic ac and for isoperimetric condition (2.2). The pressure distribution on the contour is found by solving system (1.1). For the complete formulation of the problem, it is necessary to give the method of determining p°.

3. Let p° be a constant, independent of the shape of the desired contour. Then χ is completely determined by the choice of the segment ab, whose region of influence to the right is bounded by characteristic cb. We construct the functional

$$I = \int_a^b (y^\nu p + \lambda f + \alpha^\circ L_3)\, dy + \int_b^g (y^\nu p^\circ + \lambda f^\circ)\, dy +$$
$$+ \int_b^d (\gamma_1{}^\circ L_4 + \gamma_2{}^\circ L_5 + \gamma_3{}^\circ L_6)\, d\psi + \iint_G (\mu_1{}^\circ L_1 + \mu_2{}^\circ L_2 + \mu_3{}^\circ L_3)\, d\psi\, dy$$

Here G is the region of the ψy plane, bounded by the characteristics ad and db and the vertical axis (Fig 1b); λ = const, $\alpha^\circ(y)$, $\gamma_i{}^\circ(\psi)$, and $\mu_i{}^\circ(\psi, y)$ are Lagrange multipliers. For a fixed K, because of (1.1), (1.2) and (2.2), the variations of I and of χ for any admissible variation are the same. Therefore

$$\delta\chi = \delta I = \left\{\gamma_1{}^\circ\left[\frac{\partial (v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right] + \gamma_2{}^\circ + \frac{u}{v}\gamma_3{}^\circ\right\}_d \Delta y_d + \tag{3.1}$$
$$+ \left\{\left[y^\nu (p - p^\circ) + \lambda (f - f_{x'} x') - \frac{u}{v}\alpha^\circ - \gamma_2{}^\circ\right]_- - \right.$$
$$\left. - \gamma_1{}^\circ\left[\frac{\partial (v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right]_- - \lambda f_+{}^\circ\right\}_b \Delta y_b + [(\lambda f_{x'} + \alpha^\circ - \gamma_3{}^\circ)_- - \lambda f_{x'_+}{}^\circ]_b \Delta x_b -$$

$$
\begin{aligned}
&- \{\gamma_1^\circ [\delta(v/u) - P\delta p]\}_{b-} + [(y^\nu p^\circ + \lambda f^\circ)\varphi + \lambda f_{x'}{}^\circ]_g \Delta x_g + \\
&+ \lambda \int_b^g (f_x{}^\circ - f_{x'}{}^{\circ\prime})\,\delta x\,dy + \int_a^b (U^{(0)}\delta x + U^{(1)}\delta u + U^{(2)}\delta v)\,dy + \\
&+ \int_b^d (V^{(0)}\delta x + V^{(1)}\delta u + V^{(2)}\delta v + V^{(3)}\Delta y)\,d\psi + \\
&+ \iint_G (W^{(0)}\delta x + W^{(1)}\delta u + W^{(2)}\delta v)\,d\psi\,dy
\end{aligned}
\qquad (3.1)\ \text{cont.}
$$

Here $U^{(i)}$, $V^{(i)}$ and $W^{(i)}$ are known factors of the Lagrange multipliers and other variables, $\delta\xi$ is the variation of ξ for fixed ψ and y, $\Delta\xi = \delta\xi + (\delta\xi/\delta y)\Delta y$ is the variation of ξ on the closing characteristic for fixed ψ, Δx_g is the altered abscissa of the point g, $\varphi = dy^\circ(x)/dx$, the subscripts minus (plus) indicates the quantity to the left (to the right) of the corresponding points (in the xy plane), and subscripts x and x' indicate partial derivatives with respect to x and x'. We note that although on the characteristic ad the derivatives of the flow parameters with respect to ψ and y are discontinuous, the multiplier in front of Δy_d is continuous in view of the first relation (1.2) and of the continuity of these flow parameters on ad. Defining $\mu_3{}^\circ$ in G by Equation

$$W^{(0)} \equiv -\frac{\partial \mu_3{}^\circ}{\partial y} = 0$$

and the boundary condition

$$V^{(0)} \equiv -\mu_3{}^\circ - \frac{d\gamma_3{}^\circ}{d\psi} = 0 \qquad (\text{on } db)$$

we find that $\mu_3{}^\circ(\psi, y) = \mu_3{}^\circ(\psi) = -d\gamma_3{}^\circ(\psi)\,/\,d\psi$.

Taking this into account, using the Equations

$$
\begin{gathered}
X_u - \frac{\rho u^2}{v} PX - \frac{u}{v} Y_u = 0, \qquad X_v - \frac{\rho u^2}{v} PY - \frac{u}{v} Y_v = 0 \\
X - \frac{1}{\rho v} - \frac{u}{v} Y = 0, \qquad X_u - \frac{1}{v} Y - \frac{u}{\rho v c^2} - \frac{u}{v} Y_u = 0 \\
X_v + \frac{u}{v^2} Y + \frac{c^2 - v^2}{\rho v^2 c^2} - \frac{u}{v} Y_v = 0 \\
(\zeta_u = (\partial\zeta / \partial u)_{\psi,v}, \quad \zeta_v = (\partial\zeta / \partial v)_{\psi,u})
\end{gathered}
$$

and transforming the Lagrange multipliers

$$
\begin{gathered}
\alpha(y) = \alpha^\circ - \gamma_{3b}{}^\circ, \qquad \gamma_1(\psi) = \gamma_1{}^\circ, \quad \gamma_2(\psi) = \gamma_2{}^\circ + (u/v)\gamma_3{}^\circ \qquad (3.2) \\
\gamma_3(\psi) = \gamma_3{}^\circ, \quad \mu_1(\psi, y) = \mu_1{}^\circ, \quad \mu_2(\psi, y) = \mu_2{}^\circ - \gamma_3{}^\circ, \quad \mu_3(\psi, y) = \mu_3{}^\circ
\end{gathered}
$$

we find instead of (3.1) the following:

$$\delta\chi = \delta I = \left\{\gamma_1\left[\frac{\partial(v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right] + \gamma_2\right\}_d \Delta y_d +$$

$$+ \left\{\left[y^\nu(p - p^\circ) + \lambda(f - f_{x'}x') - \frac{u}{v}\alpha - \gamma_2\right]_- -\right.$$

$$\left. - \gamma_1\left[\frac{\partial(v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right]_- - \lambda f_+^\circ\right\}_b \Delta y_b + [(\lambda f_{x'} + \alpha)_- - \lambda f_{x'+}^\circ]_b \Delta x_b -$$

$$- \{\gamma_1[\delta(v/u) - P\delta p]\}_{b-} + [(y^\nu p^\circ + \lambda f^\circ)\varphi + \lambda f_{x'}^\circ]_g \Delta x_g +$$

$$+ \lambda\int_b^g (f_x^\circ - f_{x'}^{\circ\prime})\,\delta x\,dy + \int_a^b (U^0\delta x + U^1\delta u + U^2\delta^v)\,dy +$$

$$+ \int_b^d (V^1\delta u + V^2\delta v + V^3\Delta y)\,d\psi + \iint_G (W^1\delta u + W^2\delta v)\,d\psi\,dy$$

Here U^i, V^i and W^i do not depend on γ_3 and μ_3. This means that Equation $L_6 = 0$ in general may not be introduced, while $L_3 = 0$ should be introduced in I only under the line integral sign. The remaining Lagrange multipliers are chosen so that in $\delta\chi$ there remain only the variations of the coordinates of the contour ag. Let μ_1 and μ_2 satisfy the following equations in G:

$$\begin{aligned} W^1 &\equiv -\frac{\partial\mu_1}{\partial y} - y^\nu\rho u\frac{\partial\mu_1}{\partial\psi} - \frac{u}{y^\nu\rho v c^2}\frac{\partial\mu_2}{\partial y} - \frac{1}{v}\frac{\partial\mu_2}{\partial\psi} = 0 \\ W^2 &\equiv -y^\nu\rho v\frac{\partial\mu_1}{\partial\psi} + \frac{c^2 - v^2}{y^\nu\rho c^2 v^2}\frac{\partial\mu_2}{\partial y} + \frac{u}{v^2}\frac{\partial\mu_2}{\partial\psi} = 0 \end{aligned} \tag{3.3}$$

For $w > c$, this system has the same characteristics as (1.1). Along these characteristics, the relations

$$d\mu_1 \pm \frac{\sqrt{w^2 - c^2}}{y^\nu\rho v^2 c}\,d\mu_2 = 0 \tag{3.4}$$

hold, in which the upper sign corresponds to the characteristics of the first family. The multipliers α, γ_1 and γ_2, and also the boundary conditions to determine μ_1 and μ_2, we find by equating to zero U_1 and U_2 on ab, ~ on db, and the coefficients before Δy_d and γ_{1b}. This gives

$$\mu_1 = 1, \quad \mu_2 = \alpha \qquad (\text{on } ab) \tag{3.5}$$

$$V^1 \equiv \frac{cv - u\sqrt{w^2 - c^2}}{cu^2}\frac{d\gamma_1}{d\psi} - \gamma_1\left(\frac{2\rho v}{u}P\frac{dv}{d\psi} - y^{-(\nu+1)}Q_u\right) +$$

$$+ \gamma_2 y^{-\nu}Y_u - \rho X\left(\mu_1 v + \frac{\mu_2 u^2}{y^\nu v}P\right) = 0$$

$$V^1 Y - V^2 X \equiv 2P\frac{d\gamma_1}{d\psi} + \gamma_1 P\left[\frac{4uv}{w^2}\frac{d(v/u)}{d\psi} - \nu\frac{cu + v\sqrt{w^2 - c^2}}{y^{\nu+1}\rho c w^2}\right] +$$

$$+ \gamma_2 y^{-\nu}(YY_u - XY_v) = 0 \tag{3.6}$$

$$V^3 \equiv \left\{\frac{cv - u\sqrt{w^2-c^2}}{cu^2}\frac{d\gamma_1}{d\psi} - \gamma_1\left(\frac{2\rho v}{u}P\frac{dv}{d\psi} - y^{-(\nu+1)}Q_u\right) + \gamma_2 y^{-\nu}Y_u\right\}\frac{\partial u}{\partial y} -$$
$$-\left\{\frac{cu + v\sqrt{w^2-c^2}}{cu^2}\cdot\frac{d\gamma_1}{d\psi} - \gamma_1\left(\frac{2\rho v}{u}P\frac{du}{d\psi} + y^{-(\nu+1)}Q_v\right) - \gamma_2 y^{-\nu}Y_v\right\}\frac{\partial v}{\partial y} -$$
$$-\frac{d\gamma_2}{d\psi} - \gamma_1(\nu+1)y^{-(\nu+2)}Q - \gamma_2\nu y^{-(\nu+1)}Y = 0 \qquad (\text{on } db)$$

$$\left\{\gamma_1\left[\frac{\partial(v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right] + \gamma_2\right\}_d = 0, \qquad \gamma_{1b} = 0 \tag{3.7}$$

From the linearity and homogenity of the second and third equations of (3.6) and the boundary conditions (3.7), we have

$$\gamma_1(\psi) = \gamma_2(\psi) \equiv 0$$

and the boundary condition for μ_1 and μ_2 on db, the first equation of (3.6), assumes the form

$$\mu_1 + \frac{\mu_2 u^2}{y^\nu v^2}P = 0 \qquad (\text{on } db) \tag{3.8}$$

These conditions completely determine all the Lagrange multipliers for any smooth contour ab. In accordance with these conditions we obtain

$$\delta\chi = \delta I = \{[y^\nu(p-p^\circ) + \lambda(f - f_{x'}x') - (u/v)\alpha]_- - \lambda f_+^\circ\}_b\Delta y_b +$$
$$+ [(\lambda f_{x'} + \alpha)_- - \lambda f_{x'+}^\circ]_b\Delta x_b + [(y^\nu p^\circ + \lambda f^\circ)\varphi + \lambda f_{x'}^\circ]_g\Delta x_g +$$
$$+ \lambda\int_b^g (f_x^\circ - f_{x'}^{\circ\prime})\,\delta x\,dy + \int_a^b U^0\delta x\,dy$$

If we note that the signs of δx on ab and Δy_b are arbitrary for $y_b > y_g$, while Δx_b, Δx_g and δx on bg are nonpositive (in view of the boundedness of the contour length), then we really obtain the necessary conditions for a minimum

$$\begin{gathered} U^0 \equiv \lambda(f_x - f_{x'}') - \alpha' = 0 \qquad (\text{on } ab) \\ \{[y^\nu(p-p^\circ) + \lambda(f - f_{x'}x') - (u/v)\alpha]_- - \lambda f_+^\circ\}_b = 0 \\ \lambda(f_x^\circ - f_{x'}^{\circ\prime}) \geqslant 0 \qquad (\text{on } bg) \\ [(\lambda f_{x'} + \alpha)_- - \lambda f_{x'+}^\circ]_b \leqslant 0, \qquad [(y^\nu p^\circ + \lambda f^\circ)\varphi + \lambda f_{x'}^\circ]_g \leqslant 0 \end{gathered} \tag{3.9}$$

Here the first two equations determine the shape of the optimum contour, while the remaining ones are conditions of the fact that the end wall bg is a section of boundary extremum. If the optimum contour does not have an end wall ($y_b = y_g$), then at the point g, we must have the inequality

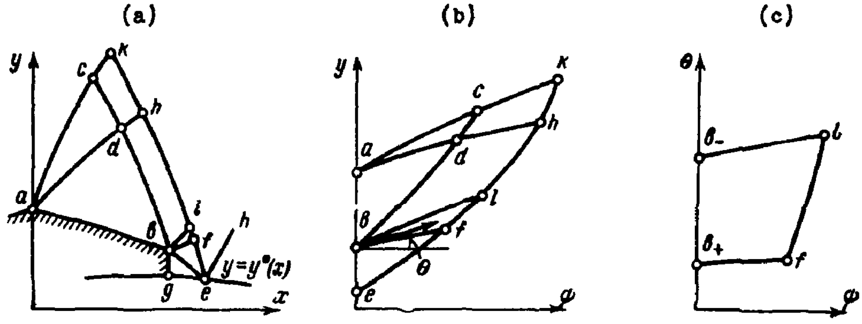
$$\{[y^\nu(p-p^\circ) + \lambda(f - f_{x'}x') - (u/v)\alpha]_- - \lambda f_+^\circ\}_g \geqslant 0 \tag{3.10}$$

These conditions agree with those found in [3], except in notations and some nonessential differences in the problem formulation. In [3] the solution was carried out without including the Equations (1.2) into the auxiliary functional. It is clear that this is a consequence of the linearity and homogeneity of the equations and boundary conditions used in determining γ_1 and γ_2.

As it follows from (3.4), (3.8) and (3.5), the values of μ_1 and μ_2 on db are determined by Formulas

$$\mu_1 = \frac{uv_b}{u_b v}\left(\frac{y_b}{y}\right)^{1/2\nu}\left(\frac{P}{P_b}\right)^{1/2}, \qquad \mu_2 = -\frac{vv_b}{uu_b}(yy_b)^{1/2\nu}(PP_b)^{-1/2}$$

4. In reality, p° is determined by the interaction of the surface $y = y^\circ(x)$ and therefore depends on the shape of the unknown contour being found. The flow configuration is given in Fig 2a, where ak, ah, bl and bf are characteristics of the first family, which bound the expansion waves kah and lbf; cb and ke are characteristics of the second family, en is a shock wave, and be is the streamline dividing the flow from the stagnation region beg. This flow region is represented in the ψy plane in Fig. 2b.



Let us assume that the pressure p° in the stagnation region is everywhere constant, that the flow at the point e behind the shock wave agrees in direction with the curve $y = y^\circ(x)$, and that of all possible flows, the one realized is that which at e satisfies the relation

$$\pi\,[p^\circ, (v/u)_{e-}, \quad \varphi_e] = 0 \tag{4.1}$$

where Π is a known function and the minus sign indicates parameters to the left of e. For the present investigation the concrete form of this function is of no value. It is only important that in accordance with (4.1)

$$[\Delta(v/u) + k_1\Delta p^\circ + k_2\Delta x]_{e-} = 0$$
$$(k_1 = [\partial\pi/\partial(v/u)](\partial\pi/\partial p^\circ)^{-1}, \quad k_2 = (\partial\pi/\partial\varphi)(d\varphi/dx)(\partial\pi/\partial p^\circ)^{-1})$$

Here, as before, $\Delta\xi$ is the variation of ξ on the closing characteristic (for fixed ψ).

The use of (4.1) introduces a change in the construction of the auxiliary functional. The region G is now bounded by the streamlines ab and be and the characteristics ah and he; the integral on ab is added to that on be, while the integral on db is replaced by that on he.

Special consideration is required for problems connected with varying y_b. Since the flow parameters at the corner undergo discontinuities, changes in the neighborhood of the point b, as we vary y_b, (with fixed ψ and y) will be finite. This renders usual methods, based on the smallness of variations, inapplicable. Thus we proceed as follows. In the integrals over regions G_0 and G_2, bounded by the contours $abih$ and bef, the integration will be carried out in ψ and y. Here the μ_i are functions of ψ and y. As to the wave ibf, we shall take a coordinate system rigidly attached to the point b. The independent variables will be chosen to be ψ and angle θ, defined as shown in Fig.2b. In the $\psi\theta$ plane (Fig.2c), the region G_1 bounded by b_+fib_-, corresponds to the ibf. The multipliers μ_i° in G_1 are taken as functions of ψ and θ, and the integrals over ψy are replaced by intervals over $\psi\theta$. By the definition of θ

$$y = y_b + \psi \tan\theta \tag{4.2}$$

and Equations (1.1) become

$$L_1^\circ \equiv \frac{\partial u}{\partial\theta} + \tan\theta\,\frac{\partial y^\nu p}{\partial\theta} - \frac{\psi}{\cos^2\theta}\,\frac{\partial y^\nu p}{\partial\psi} = 0$$

$$L_2^\circ \equiv \frac{\partial\,(y^\nu\rho v)^{-1}}{\partial\theta} - \tan\theta\,\frac{\partial\,(u\,/\,v)}{\partial\theta} + \frac{\psi}{\cos^2\theta}\,\frac{\partial\,(u\,/\,v)}{\partial\psi} = 0$$

$$L_3^\circ \equiv \frac{\partial x}{\partial\theta} - \frac{\psi}{\cos^2\theta}\,\frac{u}{v} = 0$$

In constructing the auxiliary functional, we consider in addition the possibility of discontinuities in the Lagrange multipliers on db [3]. In connection with this, the integral over G_0 will be seperated into the sum of two integrals, over G_{01} and G_{02}, respectively, each of which is a region of continuity of the μ_i°.

To this end, we take I in the form

$$I = \int_a^b (y^\nu p + \lambda f + \alpha^\circ L_3)\,dy + \int_b^e \alpha^\circ L_3 dy + \int_b^g (y^\nu p^\circ + \lambda f^\circ)\,dy +$$

$$+ \int_e^h (\gamma_1{}^\circ L_4 + \gamma_2{}^\circ L_5 + \gamma_3{}^\circ L_6)\,d\psi + \Big(\iint_{G_{01}} + \iint_{G_{02}} + \iint_{G_2}\Big)(\mu_1{}^\circ L_1 + \mu_2{}^\circ L_2 + \mu_3{}^\circ L_3)\,d\psi dy +$$

$$+ \iint_{G_1} (\mu_1{}^\circ L_1{}^\circ + \mu_2{}^\circ L_2{}^\circ + \mu_3{}^\circ L_3{}^\circ)\,d\psi\,d\theta$$

We find the first variation $\delta\chi = \delta I$. Let $\delta^\circ\xi$ be the variation of ξ in the region G_1 and on its boundaries for fixed ψ and θ. Then, according to (4.1) $\delta^\circ y = \Delta y_b$. On the boundary

$$\delta^\circ\xi = \delta\xi + \Big(\frac{\partial\xi}{\partial y}\Big)^e \Delta y_b + \Big[\Big(\frac{\partial\xi}{\partial\theta}\Big)^e - \Big(\frac{\partial\xi}{\partial\theta}\Big)^i\Big]\Delta\theta$$

Here $\delta\xi$ is the variation of ξ outside of G_1 at points of the unvaried boundary, i.e., for fixed ψ and y; the derivatives inside (outside) the centered wave region are denoted by superscripts i (e), and $\Delta\theta$ is the variation of θ on the boundary of the centered for a fixed ψ.

We require that the Lagrange multipliers be continuous on bi and bf, then determine μ_3° as was done previously, and make the substitution (3.2). As a result, the expression for $\delta\chi$, after changing back to the variables ψ and y, becomes

$$\delta\chi = \delta I = \left\{\gamma_1\left[\frac{\partial(v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right] + \gamma_2\right\}_h \Delta y_h + \left[\alpha - \left(\frac{u}{v}\alpha + \gamma_2\right)\varphi + \right.$$

$$\left. + \gamma_1 k_2\right]_{e-}\Delta x_e + \left[\frac{y_g^{\nu+1} - y_b^{\nu+1}}{\nu+1} + \gamma_{1e}(P+k_1)_{e-} - \int_b^e \mu_1 y^\nu dy\right]\Delta p^\circ +$$

$$+ \left\{[y^\nu(p - p^\circ) + \lambda(f - f_{x'}x') - (u/v)\alpha]_{b-} - [\lambda f^\circ - (u/v)\alpha]_{b+} + \right.$$

$$+ \int_{b_- f b_+}[\mu_1 d(y^\nu p) - \mu_2 d(u/v)] + \nu\iint_{G_1}\left(y^{\nu-1}p\frac{\partial\mu_1}{\partial\psi} + \frac{1}{y^{\nu+1}\rho v}\frac{\partial\mu_2}{\partial y}\right)d\psi dy\Big\}\Delta y_b +$$

$$+ [\lambda(f_{x'-} - f_{x'+}^\circ) + \alpha_- - \alpha_+]_b\Delta x_b + [(y^\nu p^\circ + \lambda f^\circ)\varphi + \lambda f_{x'}^\circ]_g\Delta x_g +$$

$$+ \lambda\int_b^g (f_x^\circ - f_{x'}^{\circ\prime})\,\delta x dy - \int_b^e [\alpha'\delta x + (\alpha - \mu_2)\delta(u/v)]\,dy +$$

$$+ \int_a^b (U^0\delta x + U^1\delta u + U^2\delta v)\,dy + \int_e^h (V^1\delta u + V^2\delta v + V^3\Delta y)\,d\psi +$$

$$+ \int_b^d (R^1\delta u + R^2\delta v)\,d\psi + \iint_{G_0+G_2}(W^1\delta u + W^2\delta v)\,d\psi\,dy +$$

$$+ \iint_{G_1}(W^1\delta^\circ u + W^2\delta^\circ v)\,d\psi\,dy$$

$$\left(R^1 = \rho X\left([\mu_1]v + \frac{u^2 P}{y^\nu v}[\mu_2]\right),\quad R^2 = R^1 Y/X\right)$$

Here $[\mu_1]$ in R^1 denotes jumps of μ_1 on the characteristic db. The choice of the Lagrange multipliers in the present case does not materially differ from that given above, and leads to the following results. The multipliers μ_1 and μ_2 are determined everywhere in G by Equations (3.3) and (3.4). The boundary conditions for the solution of these equations and the equations for determining α, γ_1 and γ_2 are: on ab, condition (3.5), on he, Equation (3.6), and on be, Equation

$$\mu_2 = \alpha = \alpha_e = \text{const}$$

The value of α_e and the boundary conditions for calculating γ_1 and γ_2 are given by the Equations

$$\left\{\gamma_1\left[\frac{\partial(v/u)}{\partial y} - P\frac{\partial p}{\partial y}\right] + \gamma_2\right\}_h = 0, \qquad \left[\alpha - \left(\frac{u}{v}\alpha + \gamma_2\right)\varphi + \gamma_1 k_2\right]_{e-} = 0$$

$$\frac{y_g^{\nu+1} - y_b^{\nu+1}}{\nu+1} + \gamma_{1e}(P+k_1)_{e-} - \int_b^e \mu_1 y^\nu dy = 0 \tag{4.3}$$

Finally, the discontinuities in the Lagrange multipliers on the characteristic db must satisfy the relation

$$[\mu_1] + \frac{u^2 P}{y^\nu v^2} [\mu_2] = 0 \tag{4.4}$$

As before, the obtained system of equations and boundary conditions determine the Lagrange multipliers for any contour ag. However, now because of the inhomogenity of the last equation in (4.3), the functions $\gamma_1 = \gamma_2 \equiv 0$ are no longer solutions.

We note that as in problems admitting a transition to characteristic contour [6], the first condition of (4.3), by virtue of the hyperbolicity of system (1.1), is satisfied at any point of the segment hl of the characteristic he, and it may be used here instead of the second or third equation in (3.6) to determine γ_1 and γ_2. In addition, one of these equations may be replaced by

$$V^3 \equiv -\frac{d\gamma_2}{d\psi} - \gamma_1 (\nu + 1) y^{-(\nu+2)} Q - \gamma_2 \nu y^{-(\nu+1)} Y +$$
$$+ \frac{y^\nu \rho u c^2}{u^2 - c^2} X \left(\frac{\rho w^2}{uv} \frac{dv}{d\psi} + \frac{\nu}{y^{\nu+1}}\right)\left(\mu_1 v + \frac{\mu_2 u^2}{y^\nu v} P\right) = 0$$

By considering the terms which remain in $\delta\chi$ after choosing the Lagrange multipliers, we get the necessary conditions for a minimum χ, which, except for the condition at the point b, are the same as (3.9). However, the conditions at the point b, after transformation with the use of (3.3), are replaced by

$$[y^\nu (p - p^\circ) + \lambda (f - f_{x'} x') - (u / v)\, \alpha]_{b-} - [\lambda f^\circ - (u / v)\, \alpha]_{b+} +$$
$$+ \int\limits_{b_-}^{b_+} [\mu_1 y^\nu \, dp - \mu_2 d\,(u / v)] = 0, \quad [\lambda (f_{x'-} - f_{x'+}{}^\circ) + \alpha_- - \alpha_+]_b \leqslant 0$$

The integral is calculated for $y = y_b$.

In the case of the absence of the wall ($y_e = y_l = y_b$), the last equation in (4.3) gives $\gamma_{1e} = 0$. From this, as previously, we get $\gamma_1 = \gamma_2 = 0$ on ke, and then from the second equation (4.3) we conclude $\alpha_e = 0$. Considering this fact, and also that for $y_b = y_e$ the characteristics he and db coincide, we obtain the same condition as in the case p° = const, including condition of the absence of the end wall (3.10). Thus, this condition holds for the general case. Earlier, this condition has been obtained by other methods in [7].

We note that extension of the domain G used in the auxiliary functional beyond the boundary of the corresponding domain of influence does not change the final results. Thus, if for p° = const, we take the region G to be an actual point, then outside the triangle abd all the Lagrange multipliers will turn out to be zero.

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